Pergamon

J. Appl. Maths Mechs, Vol. 61, No. 4, pp. 633–638, 1997

0021–8928/97 \$24.00+0.00

PII: S0021–8928(97)00081-6

THE DYNAMIC LOADING OF A PLANE ELASTIC DOMAIN WITH CONTOUR CORNER POINTS†

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(*Received* 20 *March* 1996)

The problem of the dynamic loading of plane elastic domains with an irregular boundary is considered. The energy solution of this problem in the theory of elasticity is constructed using a modified Kostrov method [1] and ideas on the extension of symmetric operators in Hilbert space. A solution of the problem of the loading of a wedge and a domain with a contour which contains m corners is obtained. © 1997 Elsevier Science Ltd. All rights reserved.

1. In the case of the plane harmonic motion of an elastic medium, the potentials of the longitudinal $\Phi(r, \theta)$ and transverse $\Psi(r, \theta)$ waves satisfy the Helmholtz equations

$$\Delta\Phi + k_1^2\Phi = 0, \quad \Delta\Psi + k_2^2\Psi = 0; \quad k_h = \omega / c_h \tag{1.1}$$

where $c_1(c_2)$ is the propagation velocity of the longitudinal (transverse) waves.

Suppose that an elastic medium, which is characterized by the Lamé constants λ and μ and a density ρ, occupies an infinite wedge-shaped domain $\Omega(-\alpha, \alpha) = \{r \geqslant 0, -\alpha \geqslant \theta \geqslant \alpha\}$. The displacements u_r, u_θ, which correspond to the specified potentials, are defined in a polar system of coordinates by the formulae

$$u_r = \frac{\partial\Phi}{\partial r} + \frac{1}{r}\frac{\partial\Psi}{\partial\theta}, \quad u_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta} - \frac{\partial\Psi}{\partial r} \tag{1.2}$$

Two types of loadings exist for which the boundary conditions are established independently for the longitudinal and transverse potentials and, consequently, they can be separated. This is the problem of the loading of a wedge, adjoining a rigid medium, with shear stresses and surface forces normal to the boundary which, according to the clamping conditions, cannot have any tangential displacements.

However, for the correct formulation of the problem, it is necessary to add to the equations for the potentials and the boundary conditions the conditions governing the behaviour of the required fields in the neighbourhood of the singular points of the domain, that is, close to the edge $r = 0$ and infinity $r \to \infty$.

A natural energy condition on the edge, which is equivalent to the requirement that the law of conservation of energy must be satisfied, can be formulated in the form

$$u = O(r^p), \; p > 0, \; \text{when } r \to 0 \tag{1.3}$$

The radiation conditions must be satisfied at infinity [2]. It has been shown [1, 3] that the requirement that the condition on the edge is satisfied does not enable one to reduce the problem completely to two acoustic cases, even with boundary conditions which permit separation of the potentials. The problem of the incidence of a plane wave on a rigid wedge inserted without friction into an infinite elastic medium has been considered in [1, 3].

Below, we consider the problem of the dynamic loading of plane domains with contour corner points. The loading was selected in such a way that the boundary conditions were set up independently for the longitudinal and transverse potentials.

We will consider the problem of the dynamic loading of a wedge. A wedge, the boundaries of which cannot have tangential displacements, by the clamping conditions, is loaded with surface forces normal to the edges, and these forces oscillate as $\exp(i\omega t)$

$$\sigma_{\theta\theta}(r, \pm\alpha) = h_\pm(r), \quad u_r(r, \pm\alpha) = 0, \quad 0 < r < \infty \tag{1.4}$$

†*Prikl. Mat. Mekh.* Vol. 61, No. 4, pp. 654–659, 1997.

A certain wave motion is therefore excited in the domain $\Omega(-\alpha, \alpha)$.
In view of the linearity, the problem splits into two independent problems, a symmetric problem and an antisymmetric problem

$$\text{(a) } \sigma_{\theta\theta}(r, \pm\alpha) = h(r), \quad \text{(b) } \sigma_{\theta\theta}(r, \pm\alpha) = \pm h(r) \tag{1.5}$$

We shall consider the symmetric formulation. In terms of the wave potentials, the boundary conditions (1.4) and (1.5) will be satisfied if it is required that

$$\Phi\big|_{\theta=\pm\alpha} = \frac{1}{\omega^2\rho}h(r), \quad \frac{\partial\Psi}{\partial\theta}\bigg|_{\theta=\pm\alpha} = -\frac{1}{\omega^2\rho}rh'(r) \tag{1.6}$$

Summarizing what has been said, we arrive at the following problem.

Problem. It is required to find the fields Φ and Ψ which, in the region $\Omega(-\alpha, \alpha)$, satisfy differential equations (1.1), boundary conditions (1.6), the condition on the edge (1.3) and the radiation conditions.

2. We will first attempt to find a solution of the problem by solving two acoustic problems for the longitudinal and transverse potentials independently, ignoring the condition on the edge.
We will seek solutions of the Helmholtz equations of the class $W^1_{2\text{loc}}(\overline{\Omega})$ when the radiation conditions are satisfied. Note that the problems are uniquely solvable in this class.
We therefore have a Dirichlet boundary-value problem for the potential Φ_a and a Neumann problem for Ψ_a. The exact solution of the problems can be obtained using a Kontorovich–Lebedev integral transform [4].
In the case of symmetric loading, the solutions of the acoustic problems can be expressed in terms of the following integrals

$$\Phi_a(r,\theta) = \frac{1}{4i\omega^2\rho}\int_0^\infty\int_{-i\infty}^{i\infty}\mu\frac{h(x)}{x}\varkappa_1(r,x,\mu)\,dx\,d\mu$$

$$\Psi_a(r,\theta) = \frac{1}{4i\omega^2\rho}\int_0^\infty\int_{-i\infty}^{i\infty}h'(x)\varkappa_2(r,x,\mu)\,dx\,d\mu$$

$$\varkappa_n(r,x,\mu) = \frac{\sin\pi\mu\, e^{-i\pi\mu}H_\mu^{(2)}(k_nr)H_\mu^{(2)}(k_hx)}{\cos\mu\alpha}, \quad n = 1,2$$

It can be shown that each of the potentials Φ_a and Ψ_a satisfies the radiation condition.
The integral representation formula for the product of MacDonald functions [5], modified for Hankel functions

$$H_\nu^{(2)}(x)H_\nu^{(2)}(y) = -\frac{2e^{\nu\pi i}}{\pi\sin\pi\nu}\int_{|\ln y/x|}^\infty J_0(\sqrt{x^2+y^2-2xy\operatorname{ch}t})\operatorname{sh}\nu t\,dt$$

$$x > 0, \quad y > 0, \quad |\operatorname{Re}\nu| < \tfrac{1}{4}$$

can be used to find the asymptotic form of the solutions close to the wedge vertex ($x \gg r$).
After some calculations, we obtain

$$\Phi_a(r,\theta) = \eta_1(\alpha)r^\xi\cos\xi\theta\int_\delta^\infty\frac{h(x)}{x}K_\xi(ik_1x)\,dx + o(1) = Ar^\xi\cos\xi\theta + o(1) \tag{2.1}$$

$$\Psi_a(r,\theta) = -\eta_2(\alpha)r^\xi\sin\xi\theta\int_\delta^\infty h'(x)K_\xi(ik_2x)\,dx + o(1) = Br^\xi\sin\xi\theta + o(1)$$

$$\eta_n(\alpha) = \frac{2\pi(ik_n)^\xi}{\alpha\omega^2 2^\xi\Gamma(\xi+1)}; \quad \xi = \frac{\pi}{2\alpha}$$

For simplicity, we shall assume that the load $h(r)$ acts at a certain distance from the wedge vertex, which ensures that the integrals in (2.1) converge.

Therefore, if the potentials Φ and Ψ are sought independently of one another, the condition on the edge will not be satisfied when $\alpha > \pi/2$ and, consequently, the displacements become infinitely large close to the wedge vertex.

Note that, in the case of an antisymmetric load, each of the potentials satisfies the condition on the edge and, consequently, the problem can be reduced to two acoustic problems.

3. In order to construct the energy solution when $\alpha > \pi/2$, we replace the requirement that $\Phi, \Psi \in W^1_{2\mathrm{loc}}(\bar{\Omega})$ by the condition $\Phi, \Psi \in L_{2\mathrm{loc}}(\bar{\Omega})$ and consider the behaviour of the required functions close to the wedge vertex. The solution of the problem for the potential $\Phi(r, \theta) \in L_{2\mathrm{loc}}(\bar{\Omega})$ at short distances r has the form

$$\Phi(r, \theta) = Ar^{\xi}\cos(\xi\theta) + b\zeta(r, \theta) + o(1) \tag{3.1}$$

where $\zeta(r, \theta)$ is a function which satisfies the Helmholtz equation in the domain Ω, the homogeneous Dirichlet condition in $\partial\Omega\backslash O$ and admits of the representation

$$\zeta(r, \theta) = r^{-\xi}\chi(r)\cos\xi\theta + Z \tag{3.2}$$

Here, $\chi(r)$ is a smooth function of the "cut-off" type, $\chi(0) = 1$ and $Z(r, \theta) \in W^1_{2\mathrm{loc}}(\bar{\Omega})$ "removes the residual"

$$\Gamma Z = g_1(r;\xi)\cos\xi\theta, \quad Z|_{\theta=\pm\alpha} = 0$$

$$g_k(r;\xi) = -r^{-\xi}[\chi''(r) + \frac{1}{r}\chi'(r)(1-2\xi) + k_j^2 r^2\chi(r)]$$

where Γ is the Helmholtz operator.

The function Z can be represented in the form [6]

$$Z(r, \theta) = cr^{\xi}\chi(r)\cos\xi\theta + R, \quad R \in W^2_{2\mathrm{loc}}(\bar{\Omega}) \tag{3.3}$$

In order to find the constant c we use a technique which is similar to that described previously in [7].

Using Green's formula for the Helmholtz operator for the functions $\zeta(r, \theta)$ and $Z(r, \theta)$ in the domain $\Omega_\delta = \Omega\backslash B_\delta$, where B_δ is a sphere of radius δ with its centre at $r = 0$, we have

$$I_\delta = \int_{\Omega_\delta} g_1(r;\xi)\zeta(r,\theta)\cos\xi\theta d\Omega_\delta = \int_{\Omega_\delta} (\zeta\Gamma Z - Z\Gamma\zeta)d\Omega_\delta = \int_{\partial\Omega_\delta} \left(\frac{\partial Z}{\partial n}\zeta - Z\frac{\partial \zeta}{\partial n}\right)d\partial\Omega_\delta$$

Substituting relation (3.3) into the last integral, we obtain

$$I_\delta = c\int_{-\alpha}^{\alpha} 2\xi\delta^{-1}\cos^2\xi\theta d\theta\delta + \delta^{\xi} \times O(1) = \pi c + o(1), \quad \delta \to 0$$

and, finally

$$c = \frac{1}{\pi}\int_{\Omega} g_1(r;\xi)\zeta(r,\theta)\cos\xi\theta d\Omega \tag{3.4}$$

Hence, the set of solutions of the problem for the potential Φ in the space $L_{2\mathrm{loc}}(\bar{\Omega})$ has the following asymptotic form close to the vertex

$$\Phi(r,\theta) = Ar^{\xi}\cos\xi\theta + b(r^{-\xi}\cos\xi\theta + cr^{\xi}\cos\xi\theta) + o(1)$$

An analogous expression is obtained for the asymptotic form of all possible solutions of the class $L_{2\mathrm{loc}}(\bar{\Omega})$ for the potential Ψ

$$\Psi(r,\theta) = Br^{\xi}\sin\xi\theta + d(r^{-\xi}\sin\xi\theta + er^{\xi}\sin\xi\theta) + o(1)$$

$$e = \frac{1}{\pi}\int_{\Omega} g_2(r;\xi)\tilde{\zeta}(r,\theta)\sin\xi\theta d\Omega$$

where $\tilde{\zeta}(r, \theta)$ is a function which satisfies the Helmholtz equation in the domain Ω, the homogeneous Neumann condition on $\partial\Omega\backslash O$ and admits of the representation

$$\tilde{\zeta}(r,\theta) = r^{-\xi}\chi(r)\sin\xi\theta + \tilde{Z}, \quad \tilde{Z}(r,\theta) \in W^1_{2\,\mathrm{loc}}(\overline{\Omega})$$

4. In order to find the solution of the problem in the theory of elasticity (to determine the constants b and d), we make use of the condition on the edge (1.3).

Close to the vertex, the components of the displacement vector can be represented in the form

$$u_r = \xi[(A + cb + B + ed)r^{\xi-1} + (b-d)r^{-\xi-1}]\cos\xi\theta + o(1)$$

$$u_\theta = \xi[(-A - cb - B - ed)r^{\xi-1} + (d-b)r^{-\xi-1}]\sin\xi\theta + o(1)$$

The constants b and d are found by equating the coefficients of the different powers of r.

5. We will now consider the problem in a polygonal domain Q which has m corners $\{\alpha_1, \ldots, \alpha_m\}$. It is assumed here that, outside a sphere of large radius, the domain coincides with a corner. As previously, we shall assume that the contour is loaded with normal surface forces which oscillate as $\exp(i\omega t)$ and that the tangential displacements in the boundary are equal to zero

$$\sigma_{\theta\theta}|_{\partial Q} = h(r), \quad u_r|_{\partial Q} = 0$$

The radiation conditions are satisfied at infinity. Using results obtained previously [6], it can be shown that, close to the ith corner, the principal terms of the asymptotic forms of the acoustic solutions have the form

$$\Phi_a = A_i r_i^{\xi_i}\cos\xi_i\theta_i + o(1), \quad \Psi_a = B_i r_i^{\xi_i}\sin\xi_i\theta_i + o(1); \quad \xi_i = \frac{\pi}{2\alpha_i} \tag{5.1}$$

where r_i, θ_i are polar coordinates with the centre at the vertex of the ith corner (henceforth there is no summation over repeated indices). In order to find the constants A_i, we make use of Green's formula for the functions ζ_i and Φ_a in the domain $Q_\delta = Q\backslash\{\cup_{k=1}^m B_{\delta_k}\}$. We have

$$\zeta_i(r) = r_i^{-\xi_i}\chi(r_i)\cos\xi_i\theta_i + Z_i =$$

$$= r_i^{-\xi_i}\chi(r_i)\cos\xi_i\theta_i + \sum_{j=1}^m c_{ij}r_j^{\xi_j}\chi(r_j)\cos\xi_j\theta_j + R_i, \quad R_i \in W^2_{2\,\mathrm{loc}}(\overline{Q}) \tag{5.2}$$

Taking account of (5.2) and the boundary conditions for Φ_a and ζ, we obtain

$$0 = \int_{\partial Q_\delta}\left(\frac{\partial\Phi_a}{\partial n}\zeta_i - \Phi_a\frac{\partial\zeta_i}{\partial n}\right)d\partial Q_\delta =$$

$$= A_i 2\xi_i\int_{-\alpha_i}^{\alpha_i}\cos^2\xi_i\theta_i d\theta_i - \frac{1}{\omega^2\rho}\int_{\partial Q_\delta}\frac{\partial\zeta_i}{\partial n}h(x)dx + o(1), \quad \delta \to 0$$

and finally, taking the limit, we have

$$A_i = \frac{1}{\pi\omega^2\rho}\int_{\partial Q}\frac{\partial\zeta_i}{\partial n}h(x)dx$$

As previously, in order to construct the energy solution of the problem in the theory of elasticity we replace the requirement that $\Phi, \Psi \in W^1_{2\mathrm{loc}}(\overline{Q})$ with the condition $\Phi, \Psi \in L_{2\mathrm{loc}}(\overline{Q})$. The solution of the problem for the potential Φ then has the form [7]

$$\Phi = \Phi_a + \sum_{i=1}^m b_i\zeta_i(x) \tag{5.3}$$

In the same way as above, we use Green's formula for the functions ζ_i and Z_j in the domain Q_δ to determine the constants c_{ij}. We obtain

$$I_\delta = \int_{Q_\delta} \zeta_i \Gamma Z_j - Z_j \Gamma \zeta_i dQ_\delta = \int_{\partial Q_\delta} \left(\frac{\partial Z_j}{\partial n} \zeta_i - Z_j \frac{\partial \zeta_i}{\partial n} \right) d\partial Q_\delta$$

We recall that Z_j satisfies the equation

$$\Gamma Z_j = -g_1(r_j;\ \xi_j)\cos\xi_j\theta_j$$

By virtue of the boundary conditions in the expressions for I_δ, only integrals along arcs of circles of small radii δ close to the corners remain. Taking account of (5.2), we obtain that the integral close to the ith corner makes the main contribution to I_δ and, consequently

$$I_\delta = \int_{Q_\delta} \Gamma Z_j \zeta_i(r) dQ_\delta =$$

$$= c_{ji} \left[\int_{-\alpha_j}^{\alpha_j} \delta\xi_j \delta^{-\xi_j} \delta^{\xi_j - 1} \cos^2 \xi_j d\theta_j + \delta^{\xi_l + 2} \times O(1) \right] = \pi c_{ji} + o(1)$$

where α_l is the largest corner differing from α_j

Hence

$$c_{ij} = \frac{1}{\pi} \int_Q \zeta_j(r) \Gamma Z_i dQ$$

The matrix $\{c_{ij}\}$ is symmetric.

In order to prove this, we use Green's formula for the Helmholtz operator for the functions ζ_i and ζ_j in the case of fixed $i \neq j$. Taking account of (5.2) and the boundary conditions we obtain that integrals along arcs of small radius δ close to the ith and jth corners make the main contribution to the integral along the contour. We have

$$0 = \int_{\partial Q_\delta} \left(\frac{\partial \zeta_j}{\partial n} \zeta_i - \zeta_j \frac{\partial \zeta_i}{\partial n} \right) d\partial Q_\delta = \pi(c_{ji} - c_{ij}) + o(1), \quad \delta \to 0$$

Taking the limit, we obtain the required equality.

The set of solutions of class $L_{2\mathrm{loc}}(\overline{Q})$ for the potential Ψ

$$\Psi = \Psi_a + \sum_{i=1}^m d_i \tilde{\zeta}_i(x) \tag{5.4}$$

can be obtained in an analogous manner. Here $\tilde{\zeta}_i$ are functions which satisfy the Helmholtz equation in the domain Q, the homogeneous Neumann condition in $\partial Q \backslash O_i$ and admit of the representation

$$\tilde{\zeta}_i(r) = r_i^{-\xi_i} \chi(r_i) \sin \xi_i \theta_i + \sum_{j=1}^m e_{ij} r_j^{\xi_j} \chi(r_j) \sin \xi_j \theta_j + \tilde{R}_i, \quad e_{ij} = \frac{1}{\pi} \int_Q \zeta_j(r) \Gamma \tilde{Z}_i dQ$$

($\{e_{ij}\}$ is a symmetric matrix).

The coefficients B_i of the asymptotic representation (5.1) are calculated using the formulae

$$B_i = -\frac{1}{\pi \omega^2 \rho} \int_{\partial Q} \tilde{\zeta}_i h'(r) r dr$$

6. In order to find the solution which possesses a finite energy, we have m relations close to the corners

$$\frac{\partial \Phi_i}{\partial r_i} + \frac{1}{r_i} \frac{\partial \Psi_i}{\partial \theta_i} = Q(r_i^p), \quad p > 0, \quad \text{when} \quad r_i \to 0$$

Substituting expressions (5.3) and (5.4) into the conditions close to the corners, we obtain $2m$ relations for determining the constants b_i, d_i

$$b_i = d_i, \quad A_i + B_i + \sum_{j=1}^{m}(c_{ji}b_j + e_{ji}d_j) = 0, \quad i = 1 \ldots m$$

or

$$b = -(C + E)^{-1}(A + B)$$

where b, A and B are m-dimensional vectors, and C and E are symmetric $m \times m$ matrices.

Note that, when there are no corners on the contour, each of the potentials satisfies the conditions on the edge and, consequently, the problem can be completely reduced to two acoustic problems.

We wish to thank S. A. Nazarov for stimulating discussions.

This research was supported financially by the Russian Foundation for Basic Research (95-01-00644a).

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Translated by E.L.S.